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INTEGRAL ESTIMATE OF THE PRESSURE IN AN INCOMPRESSIBLE MEDIUM*

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In the case of incompressible media the problems of equilibrium or of slow steady motion can, in many instances, be formulated without taking the pressure into account. The resulting "deviator" problem is usually easier to tackle, but it yields the stress deviator field τ only. The question arises in this connection of the possibility of returning to the initial formulation, i.e. of supplementing τ by a pressure field p, such that the condition of equilibrium with given volume and surface forces will hold for the stresses $\sigma = \tau + pg$ (g is the metric tensor). Since the general assertions do not, as a rule, guarantee the smoothness of τ, the problem needs special attention. In particular, an estimate of the pressure when the corresponding data on the stress deviator are available, is of interest. The estimate

$$\int_\Omega |p|^r dx \leqslant c \int_\Omega |\tau|^r dx \quad (1 < r < \infty)$$

obtained in the paper generalizes the analogous result known for $r = 2$ [1, 2]. This jusitifes the elimination of the pressure from a number of problems. Moreover, the estimate obtained can be applied directly to, for example, the pressure in perfectly plastic and viscoelastic bodies. Sect.1 gives an exact formulation of the problem and quotes examples of the cases for which it is of interest. The fundamental result is given in Sect.2 and proved in Sect.4 after establishing in Sect.3 the assertions used in the proof and concerning the fields with prescribed divergence, and re-establishment of the distribution over the derivatives. Finally, Sect.5 gives assertions facilitating the confirmation, for any problem, of the conditions under which the fundamental result was obtained.

1. Examples. Formulation of the problem. Before producing the exact formulation of the problem, we will consider several examples.

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Example 1. The problem of hydromechanics in Stokes' formulation (see e.g. /3/), in the case when the velocity is given on the boundary of the region of flow Ω and the motion takes place under the action of mass forces of density $\mathbf{F}$ reduces, when the pressure is excluded, to the problem of determining a solenoidal velocity field $\mathbf{u}$ satisfying the equation of the principle of virtual velocities

$$\nu \int_{\Omega} \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \right) e_{ij}(\mathbf{v})\, dx = \int_{\Omega} F_i v_i\, dx, \quad \forall \mathbf{v} \in \mathbf{V}^1 \tag{1.1}$$

$$e_{ij}(\mathbf{v}) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} \right); \quad i, j = 1, 2, \ldots n$$

Here ν is the coefficient of kinematic viscosity, $\mathbf{V}^1$ is the set of virtual (test) velocity fields, in this case the set of all smooth solenoidal fields finite in Ω, and x^i are Cartesian coordinates in R^n $(n = 2,3)$. Since $\operatorname{div} \mathbf{u} = 0$, $\tau_{ij} = \nu\,(\partial u_i/\partial x^j + \partial u_j/\partial x^i)$ is the stress deviator. Problem (1.1) has a solution and $\boldsymbol{\tau}$ belongs to $\mathbf{L}_2(\Omega)$ [4]. The problem of supplementing $\boldsymbol{\tau}$ by the pressure field was solved in /1/: a pressure field p belonging to $L_2(\Omega)$ can be found such that the stresses $\boldsymbol{\sigma} = \boldsymbol{\tau} + p\mathbf{g}$ satisfy the complete conditions of equilibrium

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \mathbf{e}(\mathbf{v})\, dx = \int_{\Omega} \mathbf{F}\mathbf{v}\, dx, \quad \forall \mathbf{v} \in \mathbf{V}^2$$

where $\mathbf{V}^2$ is a set of all smooth fields finite in Ω. Here p is defined, apart from an additive constant, which can be chosen so that (c is independent of $\boldsymbol{\tau}$)

$$\| p \|_{L_2(\Omega)} \leqslant c\, \| \boldsymbol{\tau} \|_{\mathbf{L}_2(\Omega)} \tag{1.2}$$

Example 2. Let a perfectly plastic medium fill the region Ω and be under the action of mass forces of volume density $\mathbf{f}$ and surface forces of density $\mathbf{q}$, specified on the part S_q of the boundary of Ω. The load reserve coefficient $(\mathbf{f}, \mathbf{q})$ can be found (see e.g. /5/) as the least upper limit of the numbers $\mu \geqslant 0$ for which the stress fields $\boldsymbol{\sigma}$ exist, not extending beyond the yield surface and equilibrating the load $(\mu\mathbf{f}, \mu\mathbf{q})$. The conditions of equilibrium have the form

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \mathbf{e}(\mathbf{v})\, dx = \mu \int_{\Omega} \mathbf{f}\mathbf{v}\, dx + \mu \int_{S_q} \mathbf{q}\mathbf{v}\, ds, \quad \forall \mathbf{v} \in \mathbf{V}^2(\Omega, S_v) \tag{1.3}$$

where the test velocity fields of which $\mathbf{V}^2(\Omega, S_v)$ is composed, are smooth and vanish near $\bar{S}_v$ (the bar denotes the closure of the set: $S_v = \partial\Omega \setminus \bar{S}_q$).

We can eliminate the spherical component of the stresses from the conditions of equilibrium, replacing $\mathbf{V}^2(\Omega, S_v)$ by the set $\mathbf{V}^1(\Omega, S_v)$ of the solenoidal velocity fields from $\mathbf{V}^2(\Omega, S_v)$, the process corresponding to the treatment of the reserve coefficient used in /6/. This is accompanied, generally speaking, by weakening of the conditions of equilibrium (due to the narrowing of the test velocity fields), and increase in the value of the reserve coefficient. Let n be this value, and let $\boldsymbol{\tau}$ be the stress deviator field on which this value is attained. Agreement between n and the value determined by the usual methods is guaranteed, provided that a pressure field p can be found such that $\boldsymbol{\sigma} = \boldsymbol{\tau} + p\mathbf{g}$ equilibrates the load $(n\mathbf{f}, n\mathbf{q})$.

Thus here we have a problem analogous to that shown in Example 1, although, unlike Example 1, the present problem must be considered for the case of mixed boundary conditions. The corresponding generalization of the results of /1/ is obtained in /2/, and the estimate (1.2) remains valid. Although this is sufficient to show the equivalence of the two formulations of the problem of perfect plasticity /2, 7/, sharpening the estimate is of interest since $\boldsymbol{\sigma} = \boldsymbol{\tau} + p\mathbf{g}$ is a real stress field in the body in question. Such a sharpening is possible, since in the plastic medium $\boldsymbol{\tau}$ belongs not only to $\mathbf{L}_2(\Omega)$, but also to $\mathbf{L}_\infty(\Omega)$.

Example 3. Consider incompressible media with a potential relation connecting the stress deviator $\boldsymbol{\tau}$ with the deformation rates $\mathbf{e}$, in which the defining relation for the stress has the form

$$\boldsymbol{\sigma} = p\mathbf{g} + \boldsymbol{\tau}(\mathbf{e}), \quad \boldsymbol{\tau}(\mathbf{e}) = \partial\varphi/\partial\mathbf{e}$$

Such media include non-linearly viscous and viscoplastic bodies, (see e.g. /8/), and in particular the media with potential

$$\varphi(\mathbf{e}) = \varphi_1(\mathbf{e}) + \varphi_2(\mathbf{e}) + \ldots + \varphi_m(\mathbf{e})$$

where φ_i is a positively homogeneous function $(s_1 > s_2 > \ldots > s_m \geqslant 1)$ of degree s_i. The solvability of the problem of the slow steady motion of such media is established in the kinematic formulation. Here $\mathbf{e}$ lies in $\mathbf{L}_{s_1}(\Omega)$ (for $s_1 > 1$) [8] and the stress deviator $\boldsymbol{\tau}$ lies correspondingly in $\mathbf{L}_{s_1'}(\Omega)$ $(s_1' = s_1/(s_1 - 1))$.

As in Example 1 and 2, the problem of introducing the pressure was not considered. If $s_1' \geqslant 2$, then the possibility of supplementing $\boldsymbol{\tau}$ by the pressure field p follows from the corresponding results for $s_1' = 2$ (since $L_{s_1'}(\Omega) \subset L_2(\Omega)$ for the bounded region Ω). As in Example 2, the sharpening of the estimate (1.2) for $s_1' > 2$ is of interest. If on the other hand $1 < s_1' < 2$, the results of /1, 2/ cannot be applied directly and additional discussion is necessary. The above examples lead naturally to the following formulation of the problem.

Let a region Ω in R^n $(n = 2, 3)$ be filled with a continuous medium acted upon by mass forces of volume density $\mathbf{f}$ and surface load of density $\mathbf{q}$, specified on the part S_q of the boundary of Ω. Further, let

$$S_v = \partial\Omega \setminus \bar{S}_q, \quad S_q = \partial\Omega \setminus \bar{S}_v$$

and let the velocity (or displacement) be defined on S_v. Let $\mathbf{s}$ $(s_{ij} = s_{ji},\ s_{ij} \in L_r(\Omega),\ 1 < r < \infty,\ i, j = 1, 2, \ldots, n)$ be any stress field equilibrating the load $(\mathbf{f}, \mathbf{q})$. Then any stress field belonging to $\mathbf{L}_r(\Omega)$ and equilibrating this load, can be written in the form $\sigma + \mathbf{s}$ where σ belongs to the set Σ_r^2 of the stress fields equilibrating the load $\mathbf{f} = 0,\ q = 0$

$$\Sigma_r^2 = \Sigma_r^2(\Omega, S_v) = \Big\{\sigma : \sigma_{ij} = \sigma_{ji}, \sigma_{ij} \in L_r(\Omega) \quad (i, j = 1, 2, \ldots, n),$$
$$\int_\Omega \sigma\cdot \mathbf{e}(\mathbf{v})\, dx = 0,\ \forall \mathbf{v} \in \mathbf{V}^2(\Omega, S_v)\Big\}$$
$$\mathbf{V}^2(\Omega, S_v) = \{\mathbf{u} \in \mathbf{C}^\infty(\bar{\Omega}) : \rho(\operatorname{supp}\mathbf{u}, \bar{S}_v) > 0\}$$

$\rho(A, B)$ is the distance between the sets A and B in R^n.

The stress deviator field equilibrating the load $(\mathbf{f}, \mathbf{q})$ less strictly, i.e. satisfying the principle of virtual velocities in which only the solenoidal velocity fields are used as the test fields, can be written in the form $\tau + \mathbf{s}^d$ where τ belongs to the set

$$\Sigma_r^1 = \Sigma_r^1(\Omega, S_v) = \Big\{\tau : \tau_{ij} = \tau_{ji}, \tau_{ii} = 0, \tau_{ij} \in$$
$$L_r(\Omega) \quad (i, j = 1, 2, \ldots, n), \int_\Omega \sigma\cdot \mathbf{e}(\mathbf{v})\, dx = 0,\ \forall \mathbf{v} \in \mathbf{V}^1(\Omega, S_v)\Big\}$$
$$\mathbf{V}^1(\Omega, S_v) = \{\mathbf{u} \in \mathbf{C}^\infty(\bar{\Omega}):\ \operatorname{div}\mathbf{u} = 0,\ \rho(\operatorname{supp}\mathbf{u},\ \bar{S}_v) > 0\}$$

(the superscript d denotes the deviator component of the tensor).

We require to establish that for any stress field τ belonging to $\Sigma_r^1(\Omega, S_v)$, i.e. equilibrating the given load in a weaker sense, a pressure field p can be found such that $\tau + pg$ belongs to $\Sigma_r^2(\Omega, S_v)$, i.e. equilibrates the load in the usual sense. In addition, we must obtain an estimate for $\|p\|_{L_r(\Omega)}$. We note that when the force conditions $(S_q \neq \varnothing)$ are given on the part $\partial\Omega$, the pressure field for a given τ is determined uniquely, while if the kinematic conditions $(\partial\Omega = S_v)$ are specified on the whole boundary, p is defined apart from an arbitrary additive constant.

Note. In determining the sets of selfequilibrated stresses Σ_r^1, Σ_r^2 we have used the virtual velocity fields vanishing near $\bar{S}_v$. Lemma 5.3 (see below) implies that this does not lead to widening of Σ_r^1, Σ_r^2 as compared with the sets determined in the same manner but for the test velocities which vanish only on S_v.

2. Formulation of the fundamental result.

We will introduce the following notation:

$$\mathbf{U}_r^1 = \mathbf{U}_r^1(\Omega, S_v) = [\{\mathbf{u} \in \mathbf{C}^\infty(\bar{\Omega}): \operatorname{div}\mathbf{u} = 0,\ \rho(\operatorname{supp}\mathbf{u},\ S_v) > 0\}]_{\mathbf{W}^1_r(\Omega)} \tag{2.1}$$
$$\hat{\mathbf{U}}_r^1 = \hat{\mathbf{U}}_r^1(\Omega, S_v) = \{\mathbf{u} \in \mathbf{W}_r^1(\Omega) : \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}|_{S_v} = 0\}$$

(square brackets denote the closure of the set in the corresponding space).

Theorem 2.1. Let Ω be a bounded Lipshitz region and $1 < r < \infty$. Then for any τ belonging to $\Sigma_r^1 = \Sigma_r^1(\Omega, \partial\Omega)$ a pressure field p can be found such that $\tau + pg$ belongs to $\Sigma_r^2 = \Sigma_r^2(\Omega, \partial\Omega)$ and (c is independent of τ)

$$\|p\|_{L_r(\Omega)} \leqslant c\,\|\tau\|_{\mathbf{L}_r(\Omega)}$$

Theorem 2.2. Let Ω be a bounded region of class C^1. Then the force conditions $S_q \neq \varnothing$ and $1 < r' < \infty$ are specified on at least a part of its boundary. Let Ω and S_v be such that

$$\mathbf{U}_{r'}^1(\Omega, S_v) = \hat{\mathbf{U}}_{r'}^1(\Omega, S_v) \tag{2.2}$$

Then for any τ belonging to $\Sigma_r^1 = \Sigma_r^1(\Omega, S_v)$ a pressure field p can be found such that $\tau + pg$ belongs to $\Sigma_r^2 = \Sigma_r^2(\Omega, S_v)$ and (c is independent of τ)

$$\|p\|_{L_r(\Omega)} \leqslant c\,\|\tau\|_{\mathbf{L}_r(\Omega)}, \quad \frac{1}{r} + \frac{1}{r'} = 1$$

Notes. 1°. Assumption (2.2) is obviously formulated unconstructively. Below in Sect.5

we shall give simple sufficient conditions guaranteeing the validity of (2.2) and covering a wide class of Ω and S_v for $1 < r < \infty$.

2°. The question arises of how arbitrary the choice of the pressure field p can be ensuring satisfaction of the equilibrium conditions for $\sigma = \tau + pg$ when $\tau \in \Sigma_r^1$, if p is not bound by the requirement $p \in L_r(\Omega)$. Since the conditions of equilibrium consist, in particular, of the relation $\partial\sigma_{ij}/\partial x^j = 0$ in the sense of distributions, the arbitrariness in the choice of such p in the class of distributions is exhausted by the addition of an arbitrary constant if $\partial\Omega = S_v$. If on the other hand $\partial\Omega \neq S_v$ $(S_q \neq \varnothing)$, then p is uniquely defined.

Theorems 2.1 and 2.2. answer the question posed in Sect.1. In particular, they allow us to assert that, if the given load can in general be balanced by the stress field with bounded deviator and pressure, summable in degree $r > 1$, then any real stress field equilibrating this load in a perfectly plastic body also has the pressure summable in the same degree. These theorems also establish the existence and give an estimate of the pressure in the case of slow steady motions of the medium discussed in Example 3.

Generally speaking, if for some incompressible medium the pressure enters the conditions of equilibrium only, then it can be eliminated from the formulation of the problem. This is achieved by replacing the set $V^2(\Omega, S_v)$ of test velocity fields by the set $V^1(\Omega, S_v)$. If the "deviator" problem obtained in this manner has a solution with stress deviator summable in degree r (and the load given can be equilibrated, in general, by any stress field summable in degree r), then Theorems 2.1 and 2.2 imply that the initial problem is also solvable and the pressure is summable in the same degree r.

3. Vector fields with prescribed divergence and reestablishment of the distribution over the derivatives. Let us establish the validity of certain assertions used in proving Theorems 2.1 and 2.2 and in deriving the sufficient conditions for $\mathbf{U}_r^1(\Omega, S_v)$ and $\mathbf{U}_r^{\wedge 1}(\Omega, S_v)$ to be identical. The conditions generalize the results known for $r = 2$ [4, 1] to the case $1 < r < \infty$. Below, $\mathbf{v}$ will denote the unit normal to the region boundary, and we shall also use the notation

$$L_r^\wedge(\Omega) = \left\{u \in L_r(\Omega) : \int_\Omega u\,dx = 0\right\}$$

As usual, $W_r^{1/r'}(\partial\Omega)$ is the space of traces on $\partial\Omega$ of functions belonging to $W_r^1(\Omega)$ [9, 10], $r' = r/(r-1)$, $1 < r < \infty$. Finally, we denote (different) constants by c.

Lemma 3.1. Let Ω be a bounded region belonging to the class C^3, $1 < r < \infty$. Then for any α from $\mathbf{W}_r^{1/r'}(\partial\Omega)$ such that

$$\int_{\partial\Omega} \alpha\mathbf{v}\,ds = 0$$

the problem (c is independent of α)

$$\mathbf{a} \in \mathbf{W}_r^1(\Omega), \quad \operatorname{div}\mathbf{a} = 0, \quad \mathbf{a}|_{\partial\Omega} = \alpha, \quad \|\mathbf{a}\|_{\mathbf{W}_r^1(\Omega)} \leqslant c\,\|\alpha\|_{\mathbf{W}_r^{1/r'}(\partial\Omega)} \tag{3.1}$$

has a solution. The proof is analogous to that of the corresponding assertion for $r = 2$ (/4/, Sect.2). The estimate for $\|\mathbf{a}\|_{\mathbf{W}_r^1(\Omega)}$ follows from the estimate

$$\|u\|_{W_r^2(\Omega)} \leqslant c\,\|f\|_{L_r(\Omega)}$$

of the solution to the Neumann problem

$$\Delta u = f \text{ in } \Omega, \ \frac{\partial u}{\partial \mathbf{v}}\Big|_{\partial\Omega} = 0, \ \int_\Omega u\,dx = 0 \quad (f \in L_r^\wedge(\Omega))$$

The estimate, in turn, follows from the inequality /11/

$$\|u\|_{W_r^2(\Omega)} \leqslant c\left(\|\Delta u\|_{L_r(\Omega)} + \left\|\frac{\partial u}{\partial \mathbf{v}}\right\|_{W_r^{1/r'}(\partial\Omega)} + \|u\|_{L_r(\Omega)}\right)$$

and the Petric lemma (Lemma 3 of /12/). From Lemma 3.1, just as in /1/ for $r = 2$, we have

Lemma 3.2. Let Ω be a bounded region of class C^3; $1 < r < \infty$. Then for any φ from $L_r^\wedge(\Omega)$ the problem (c is independent of φ)

$$\mathbf{a} \in W_r^{\circ 1}(\Omega), \quad \operatorname{div}\mathbf{a} = \varphi, \quad \|\mathbf{a}\|_{W_r^1(\Omega)} \leqslant c\,\|\varphi\|_{L_r(\Omega)} \tag{3.2}$$

has a solution.

Next we consider the following problem of restoring the distribution over its derivatives: how regular will the distribution of p be, if its derivatives $\partial p/\partial x^i$ belong to the space $W_r^{-1}(\Omega) = (W_{r'}^{\circ 1}(\Omega))'$.

Theorem 3.1. Let Ω be a bounded Lipshits region ($1 < r < \infty$). If p and X_i are distributions on Ω

$$\frac{\partial p}{\partial x^i} = X_i, \quad X_i \in W_r^{-1}(\Omega)$$

then p belongs to $L_r(\Omega)$ and a constant p_0 can be found such that (c is independent of p)

$$\| p - p_0 \|_{L_r(\Omega)} \leqslant c \| \mathbf{X} \|_{W_r^{-1}(\Omega)} \tag{3.3}$$

Proof. First we note that it is sufficient to prove the theorem for a star-like region (region G is called star-like with respect to the point x_* if the $\varkappa > 1$ -fold contraction of G with the centre at x_* maps G onto its strictly interior subregion). Indeed, the Lipshits region Ω can be written in the form

$$\Omega = \bigcup_{\alpha=1}^{N} \Omega_\alpha$$

where Ω_α are star-like. If the theorem holds for a star-like region, then

$$p|_{\Omega_\alpha} \in L_r(\Omega_\alpha), \quad \| p|_{\Omega_\alpha} - p_{0\alpha} \|_{L_r(\Omega_\alpha)} \leqslant c \| \mathbf{X} \|_{W_r^{-1}(\Omega)}$$

In this case we can choose p_0 so that the estimate (3.3) holds. To do this it is sufficient to take /1/

$$p_0 = \Big(\sum_{\alpha=1}^{N} \operatorname{mes} \Omega_\alpha \Big)^{-1} \sum_{\alpha=1}^{N} \int_{\Omega_\alpha} p \, dx$$

Thus below we can assume that Ω is a star-like region (relative to zero).

According to the "structural" theorem for $W_r^{-1}(\Omega)$ [13] X_i can be written in the form (c is independent of X)

$$X_i = \frac{\partial \sigma_{ik}}{\partial x^k} + f_i, \quad \sigma_{ik} \in L_r(\Omega), \quad f_i \in L_r(\Omega) \tag{3.4}$$
$$\| \sigma \|_{L_r(\Omega)} \leqslant c \| \mathbf{X} \|_{W_r^{-1}(\Omega)}, \quad \| \mathbf{f} \|_{L_r(\Omega)} \leqslant c \| \mathbf{X} \|_{W_r^{-1}(\Omega)}$$

Let us consider in $\lambda\Omega$ ($0 < \lambda_0 < \lambda < 1$) the averages p^h, $\sigma_{ik}{}^h$, $f_i{}^h$ of the quantities p, σ_{ik}, f_i for sufficiently small diameter h of the support of the averaging kernel. The smooth functions p^h, $\sigma_{ik}{}^h$, $f_i{}^h$ in $\lambda\Omega$ satisfy the relation

$$\frac{\partial p^h}{\partial x^i} = \frac{\partial \sigma_{ik}^h}{\partial x^k} + f_i^h$$

We can apply to these functions the analog of Lemma 2.4 /1/, which is proved for $1 < r < \infty$ just as for $r = 2$ [1] (in the proof we use the integral representation of the functions, which also applies in the case $r \neq 2$ and Lemma 3.2). According to this assertion for the function

$$p^{\wedge h} = p^h - [\operatorname{mes}(\lambda_0 \Omega)]^{-1} \int_{\lambda_0 \Omega} p^h \, dx \tag{3.5}$$

the following estimate holds:

$$\| p^{\wedge h} \|_{L_r(\lambda\Omega)} \leqslant c [1 + (\lambda/\lambda_0)^{n/r}] (\| \sigma \|_{L_r(\lambda\Omega)} + \| \mathbf{f} \|_{L_r(\lambda\Omega)}) \tag{3.6}$$

where c depends only on $\lambda\Omega$, r, n.

Since σ^h, $\mathbf{f}^h$ converges as $h \to 0$ in $\mathbf{L}_r(\Omega)$ to σ, $\mathbf{f}$, therefore according to the previous inequality $p^{\wedge h}$ also converges to some $p^\wedge \in L_r(\Omega)$. Here in $\lambda\Omega$ we have

$$\frac{\partial p^\wedge}{\partial x^i} = \frac{\partial \sigma_{ik}}{\partial x^k} + f_i$$

and an inequality analogous to (3.6) occurs. Using the fact that Ω and $\lambda\Omega$ are similar, we can write this inequality in the form (c_Ω depends only on Ω, r, n)

$$\| p^\wedge \|_{L_r(\lambda\Omega)} \leqslant c_\Omega [1 + (\lambda/\lambda_0)^{n/r}] (\| \sigma \|_{L_r(\lambda\Omega)} + \lambda \| \mathbf{f} \|_{L_r(\lambda\Omega)}) \tag{3.7}$$

By virtue of (3.5), $p^\wedge$ satisfies the condition

$$\int_{\lambda_0\Omega} p^\wedge \, dx = 0$$

Therefore the functions $p^{\wedge}$, defined for various $\lambda(\lambda_0<\lambda<1)$, are identical at the intersections of their regions of definition and can thus be regarded as contractions on $\lambda\Omega$ of some function P defined on Ω . The following relations obviously hold for this function:

$$\frac{\partial P}{\partial x^i}=\frac{\partial \sigma_{ik}}{\partial x^k}+f_i=X_i \tag{3.8}$$

and, in accordance with (3.7), the following estimate holds:

$$\| P \|_{L_r(\Omega)} \leqslant c_\Omega [1+\lambda_0^{-n/r}] (\| \boldsymbol{\sigma} \|_{L_r(\Omega)} + \| \mathbf{f} \|_{L_r(\Omega)}) \tag{3.9}$$

We note that by virtue of (3.8) P may differ from p only by a constant. Using (3.4) we obtain from the inequality (3.9) the estimate (3.3), and this completes the proof. We will use Theorem 3.1 to establish the solvability of problem (3.2) for a much wider class of regions than in Lemma 3.2.

Lemma 3.3. Let Ω be a bounded Lipshits region; $1<r<\infty$. Then problem (3.2) has a solution for any φ from $L_r^{\wedge}(\Omega)$ (c is independent of φ)

Proof. First we confirm that if for given φ

$$\mathbf{b} \in \mathbf{W}_r^{-1}(\Omega),\ \operatorname{div} \mathbf{b}=\varphi \tag{3.10}$$

can be found, then we can also find $\mathbf{a}$ satisfying (3.2). Indeed, in this case

$$\mathbf{U}_\varphi \equiv \{\mathbf{u} \in \mathbf{W}_r^{\circ 1}(\Omega):\ \operatorname{div} \mathbf{u}=\varphi\} \neq \varnothing$$

is a closed subspace in $\mathbf{W}_r^1(\Omega)$, in which case we can obtain /14/ the lower bound

$$\inf \{\| \mathbf{u} \|_{\mathbf{W}_r^1(\Omega)} : \mathbf{u} \in \mathbf{U}_\varphi\} = \| \mathbf{a} \|_{\mathbf{W}_r^1(\Omega)},\quad \mathbf{a} \in \mathbf{U}_\varphi$$

From the necessary condition for a minimum [15] $\partial \| \mathbf{a} \| \cap (\mathbf{U}_0)^\circ \neq \varnothing$, i.e. $\mathbf{g} \in (\mathbf{W}_r^1(\Omega))'$ can be found such that

$$\| \mathbf{g} \|_{(\mathbf{W}_r^1(\Omega))'}=1,\quad \langle \mathbf{g}, \mathbf{a}\rangle = \| \mathbf{a} \|_{\mathbf{W}_r^1(\Omega)},\quad \langle \mathbf{g}, \mathbf{u}\rangle=0,\quad \forall \mathbf{u} \in \mathbf{U}_0 \tag{3.11}$$

From the last property it follows /16/ that $\mathbf{g}=\operatorname{grad}\psi$ and according to the theorem 3.1 $\psi \in L_{r'}(\Omega)$, $\|\psi\|_{L_{r'}(\Omega)} \leqslant c$ (the first relation of (3.11) was used). Then

$$\| \mathbf{a} \|_{\mathbf{W}_r^1(\Omega)} = \langle \operatorname{grad}\psi, \mathbf{a}\rangle = -\langle \psi, \operatorname{div}\mathbf{a}\rangle \leqslant \|\psi\|_{L_{r'}(\Omega)} \|\varphi\|_{L_r(\Omega)} \leqslant c \|\varphi\|_{L_r(\Omega)}$$

which it was required to prove.

Let us now consider the operator $\operatorname{div}: \mathbf{W}_r^{\circ 1}(\Omega) \to L_r^{\wedge}(\Omega)$. Let

$$\mathbf{v}_n \in \mathbf{W}_r^{\circ 1}(\Omega),\ \varphi_n=\operatorname{div}\mathbf{v}_n,\ \varphi_n \to \varphi \text{ in } L_r(\Omega)\ (n\to\infty)$$

From what was proved we can find $\mathbf{a}_n \in W_r^{\circ 1}(\Omega)$ with $\operatorname{div}\mathbf{a}_n=\varphi_n$ and $\|\mathbf{a}_n\|_{W_r^1(\Omega)} \leqslant c\|\varphi\|_{L_r(\Omega)}$. Then we can separate from the sequence $\{\mathbf{a}_n\}$ a subsequence weakly converging in $\mathbf{W}_r^1(\Omega)$. Let $\mathbf{a}$ be its limit. Then $\mathbf{a} \in \mathbf{W}_r^{\circ 1}(\Omega)$ and $\operatorname{div}\mathbf{a}=\varphi$. This means that the set of values of the operator in question is closed. In this case problem (3.10) will have a solution for any φ from $L_r^{\wedge}(\Omega)$, if and only if the equation $\operatorname{div}^T \mathbf{f}=0$ has only a null solution (the index T denotes a conjugate operator). Since $(L_r^{\wedge})'$ is identical as a vector space and has equivalent norm with $\mathbf{L}_{r'}^{\wedge}$ and $\operatorname{div}^T=-\operatorname{grad}$, it follows that the equation has the form $\operatorname{grad}\mathbf{f}=0,\ \mathbf{f}\in\mathbf{L}_{r'}^{\wedge}$ and has clearly only a null solution. This means that problem (3.10) has a solution and hence, as was shown above, also (3.2). This proves the lemma. An analogous assertion was proved for $r=2$ for a some what wider class of regions in /1/.

4. Proof of Theorems 2.1 and 2.2. When $r=2$ the corresponding assertions were proved in /1, 2/. The proofs for $1<r<\infty$ are carried out by the same scheme, using Theorem 3.1 and Lemma 3.3 proved above.

Proof of Theorem 2.1. Since $\boldsymbol{\tau} \in \Sigma_r^1(\Omega, \partial\Omega)$, it follows that $\langle \partial\tau_{ij}/\partial x^j, v_i\rangle=0$ for any $\mathbf{v}$ from $\mathbf{D}(\Omega)$ with $\operatorname{div}\mathbf{v}=0$. Then a distribution /16/ of p on Ω can be found such that

$$\frac{\partial p}{\partial x^i}+\frac{\partial \tau_{ij}}{\partial x^j}=0 \tag{4.1}$$

We note that $\partial\tau_{ij}/\partial x^j$ belongs to the space $W_r^{-1}(\Omega)$ conjugate to $W_r^{\circ 1}(\Omega)$, and

$$\left\| \frac{\partial \tau_{ij}}{\partial x^j} \right\|_{W_r^{-1}(\Omega)} \leqslant c \|\tau_{ij}\|_{L_r(\Omega)}$$

By Theorem 3.1, p belongs to $L_r(\Omega)$ and can be chosen so that $\|p\|_{L_r(\Omega)} \leqslant c\|\boldsymbol{\tau}\|_{L_r(\Omega)}$. Then $\boldsymbol{\sigma}=\boldsymbol{\tau}+p\mathbf{g} \in \mathbf{L}_r(\Omega)$ and by virtue of (4.1) we have

$$\langle \sigma_{ij},\ \partial v_i/\partial x^j\rangle = 0,\ \ \forall \mathbf{v} \in \mathbf{D}(\Omega) \tag{4.2}$$

Since when $\partial\Omega=S_v$ the set of test velocity fields $\mathbf{V}^2(\Omega, S_v)$ is identical with $\mathbf{D}(\Omega)$, (4.2) means that $\sigma \in \Sigma_r^2(\Omega, S_v)$, which completes the proof.

Proof of Theorem 2.2. First we will establish, as before, the existence of a pressure field $p^0 \in L_r(\Omega)$ such that

$$\frac{\partial \sigma_{ij}^\circ}{\partial x^j} = 0 \quad (\sigma^\circ = \tau + p^\circ g), \quad \| p^\circ \|_{L_r(\Omega)} \leqslant c \| \tau \|_{L_r(\Omega)} \tag{4.3}$$

Further, we can collect the vectors $\sigma_{(i)}^\circ$ with components $(\sigma_{i1}^\circ, \sigma_{i2}^\circ, \ldots, \sigma_{in}^\circ)$ and consider the traces $\gamma_{(i)}$ of their normal components on $\partial\Omega$. As in /17/ we confirm, that for any $\mathbf{v}$ from $\mathbf{U}_{r'}^1$ we have

$$\langle \gamma, \mathbf{v} |_{\partial\Omega} \rangle = 0, \ \gamma = (\gamma_{(1)}, \gamma_{(2)}, \ldots, \gamma_{(n)}) \tag{4.4}$$

Here $\mathbf{v}$ belongs to $W_{r'}^{1/r}(\partial\Omega)$ [9, 10] and γ to the space conjugate to it. This, together with (4.4), follows from (4.3) and the following assertion.

Lemma 4.1. Let $u_\nu = \mathbf{u}|_{\partial\Omega}\mathbf{v}$ be the trace on $\partial\Omega$ of the normal component of the field $\mathbf{u}$ smooth in $\bar{\Omega}$, which can be considered as a continuous linear functional on $W_{r'}^{1/r}(\partial\Omega)$ acting according to the formula

$$\langle u_\nu, w \rangle = \int_{\partial\Omega} u_\nu w \, ds \quad (w \in W_{r'}^{1/r}(\partial\Omega))$$

If Ω is a bounded region of class C^1, the mapping $\mathbf{u} \to u_\nu$ can be continued to the continuous linear mapping from the space

$$\mathbf{K}_r(\Omega) = \{\mathbf{u} \in \mathbf{L}_r(\Omega) : \operatorname{div} \mathbf{u} \in L_r(\Omega)\}, \quad \| \mathbf{u} \|_{\mathbf{K}_r(\Omega)}^r = \| \mathbf{u} \|_{\mathbf{L}_r(\Omega)}^r + \| \operatorname{div} \mathbf{u} \|_{L_r(\Omega)}^r$$

into the space $(W_{r'}^{1/r}(\partial\Omega))'$. The following formula holds for any w from $W_{r'}^1(\Omega)$ and any $\mathbf{u}$ from $\mathbf{K}_r(\Omega)$

$$\int_\Omega w \operatorname{div} \mathbf{u} \, dx = -\int_\Omega \mathbf{u} \operatorname{grad} w \, dx + \langle u_\nu, w|_{\partial\Omega} \rangle \tag{4.5}$$

Lemma 4.1 is proved in the same manner as the analogous assertion in /17/. We note in addition that the last term on the right-hand side of (4.5) can be found as

$$\lim_{n\to\infty} \int_{\partial\Omega} u_\nu^{(n)} w |_{\partial\Omega} \, ds$$

where $\{\mathbf{u}^{(n)}\}$ denotes any sequence of smooth functions converging to $\mathbf{u}$ in $\mathbf{K}_r(\Omega)$.

If now $\mathbf{u}$ is any field belonging to

$$\mathbf{U}_{r'}^2 = \mathbf{U}_{r'}^2(\Omega, S_v) = [\mathbf{V}^2(\Omega, S_v)]_{\mathbf{W}_{r'}^1(\Omega)}$$

then using Lemma 3.3 we find, as in /17/, that

$$\langle \gamma, \mathbf{u}|_{\partial\Omega} \rangle = c_0 \int_{\partial\Omega} \mathbf{u}\mathbf{v} \, ds, \quad c_0 = \langle \gamma, \mathbf{u}_0|_{\partial\Omega} \rangle$$

where $\mathbf{u}_0$ is any field from $\mathbf{U}_{r'}^2$ for which

$$\mathbf{u}_0|_{S_v} = 0, \quad \int_{\partial\Omega} \mathbf{u}_0 \mathbf{v} \, ds = 1$$

Then putting $p = p^0 - c_0$ we find from (4.5) using (4.3), that for $\sigma = \tau + pg$

$$\int_\Omega \sigma_{ij} \frac{\partial u_i}{\partial x^j} dx = \langle \gamma, \mathbf{u}|_{\partial\Omega} \rangle - \langle c_0 \mathbf{v}, \mathbf{u}|_{\partial\Omega} \rangle = 0, \quad \forall \mathbf{u} \in \mathbf{U}_{r'}^2$$

and hence σ belongs to Σ_r^2 . Further, since

$$|c_0| \leqslant c \| \gamma \|_{(\mathbf{W}_{r'}^{1/r}(\partial\Omega))'}$$

and the mapping of the trace is continuous according to Lemma 4.1, we have $|c_0| \leqslant c \| \sigma^\circ \|_{\mathbf{L}_r(\Omega)}$, which together with inequality (4.3) yields the estimate

$$\| p \|_{L_r(\Omega)} \leqslant c \| \tau \|_{\mathbf{L}_r(\Omega)}$$

which completes the proof.

5. The sufficient conditions for U_r^1 and $U_r^{\sim 1}$ to be identical.

The formulation of Theorem 2.1 includes the assumption that the sets $U_r^1(\Omega, S_v), U_r^{\sim 1}(\Omega, S_v)$ (see (2.1)) are identical, or in other words, the assumption that the smooth solenoidal fields, vanishing in the neighbourhood of S_v , are dense in the subspace of solenoidal fields with null trace on S_v belonging to $W_r^1(\Omega)$. Below we will give the conditions which ensure that this assumption holds. We use the congruence of the corresponding sets

$$U_r^1(\Omega, \partial\Omega) = [\{u \in D(\Omega): \operatorname{div} u = 0\}]_{W_r^1(\Omega)}$$

$$U_r^{\frown 1}(\Omega, \partial\Omega) = \{u \in W_r^1(\Omega): \operatorname{div} u = 0, \ u|_{\partial\Omega} = 0\}$$

with $S_v = \partial\Omega$ [1, 18] (which follows also from Theorem 3.1). We introduce the notation

$$U_r^2 = U_r^2(\Omega, S_v) = [\{u \in C^\infty(\overline{\Omega}): \rho(\operatorname{supp} u, \ \bar{S}_v) > 0\}]_{W_r^1(\Omega)}$$

$$U_r^{\frown 2} = U_r^{\frown 2}(\Omega, S_v) = \{u \in W_r^1(\Omega): \operatorname{div} u = 0, \ u|_{S_v} = 0\}$$

The following lemma is proved in the same way as Theorem 3.1 /17/.

Lemma 5.1. Let 1) Ω be a bounded, strictly Lipshits region; 2) $\bar{S}_q$ can be regarded as a common part of the boundaries of the intersecting regions Ω and Ω' where Ω' is Lipshits; 3) region G contains Ω and Ω' and is such that $G = \Omega \cup \Omega'$ is Lipshits; 4) for some $r\ (1 < r < \infty)$ $U_r^2(\Omega, S_v) = U_r^{\frown 2}(\Omega, S_v)$. Then $U_r^1(\Omega, S_v) = U_r^{\frown 1}(\Omega, S_v)$. The sufficient conditions for assumption 4 to hold are given by Lemma 5.3, and we shall give another assertion indicating the simple conditions for U_r^1 and $U_r^{\frown 1}$ to be identical.

Let Ω and S_v be regularly assembled from the regions Ω' and Ω'' and parts of their boundaries S_v' and S_v'' respectively (for the definition of regular assembly see e.g. /17/, Sect.3). Then the intersection $\Omega' \cap \Omega''$ decomposes into a finite number of mutually unconnected regions $\Omega_i\ (i = 1, 2, \ldots, N)$. Let us also write $S' = \bigcup_i (\partial\Omega_i \cap \Omega'')$ and $S'' = \bigcup_i (\partial\Omega_i \cap \Omega')$.

Lemma 5.2. Let 1) Ω and S_v be regularly assembled from Ω', Ω'' and S_v', S_v'' respectively; 2) $\Omega, \Omega_i\ (i = 1, 2, \ldots, N)$ are Lipshits regions and the boundary of Ω_i contains a non-empty set open in $\partial\Omega_i$ and lying in S_q; 3) the following relations hold for some $r\ (1 < r < \infty)$:

$$U_r^1(\Omega', S_v \cup S') = U_r^{\frown 1}(\Omega', S_v' \cup S'), \quad U_r^1(\Omega'', S_v'' \cup S'') = U_r^{\frown 1}(\Omega'', S_v'' \cup S'')$$

Then $U_r^1(\Omega, S_v) = U_r^{\frown 1}(\Omega, S_v)$. Its proof is analogous to that of Theorem 3.2 /17/ where we should use Lemma 3.3 in place of the corresponding assertion for $r = 2$.

Let us now return to the problem of the identical form of the sets U_r^2 and $U_r^{\frown 2}$ (see the conditions of Lemma 5.1).

Lemma 5.3. Let Ω be a bounded region of class C^1 and S_v a regular part of its boundary. Then $U_r^2(\Omega, S_v) = U_r^{\frown 2}(\Omega, S_v)$. The proof and the concept of regularity are the same as in Lemma 2.1 /17/. We merely note that the sufficient condition for the regularity of the set S_v is, that it (not necessarily connected) should be bounded by a finite number of Lipshits curves.

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THE INVERSE PROBLEM OF ACOUSTIC-WAVE SCATTERING FOR THIN ACOUSTICALLY RIGID BODIES*

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A description of an analytical algorithm for determining the shape of the scatterer with acoustically rigid walls is given for low-frequency scattering of plane acoustic waves. It is assumed that the amplitude of the scattering is known in a direction close to the direction of backward scattering, specified on a discrete set of probing wave numbers. The incident wavelength is of the order of the length of the scatterer and is much greater than its thickness.

A problem similar to this was discussed in /1/, where, however, it was assumed that the amplitude of the scattered plane waves is known on the surface of the unit sphere, and in a continuous spectrum of fairly small wave numbers. It was also assumed that there is a priori information available on the minimum radius of the sphere containing the scatterer inside it, and that the constant which limits the potential gradient of the velocity of the liquid outside the sphere is known for the problem of determining the streamlines of a vortex-free liquid.

Suppose that an absolutely rigid body D_* with a boundary S, described by the equation

$$r = \varepsilon F(t, \varphi),\ 0 \leqslant t \leqslant a,\ \ 0 \leqslant \varphi \leqslant 2\pi,\ F(0, \varphi) = F(a, \varphi) = 0 \tag{1}$$

is situated in a space filled with an acoustic medium, where r, φ, t are cylindrical coordinates with origin at the point O, and $\varepsilon > 0$ is a small parameter. The function $F^2(t, \varphi)$ is assumed to be integrable with a square on the surface of the unit cylinder $G = \{0 \leqslant t \leqslant a,\ 0 \leqslant \varphi \leqslant 2\pi, r = 1\}$. We will assume that a plane wave $u_n(\mathbf{x}) = A_0 \exp[ik(\mathbf{l}, \mathbf{x})]$ is incident from infinity on the body D_* (here and henceforth the time factor $\exp[-ika\tau]$ is omitted), where $\mathbf{l} = (l_1, l_2, l_3)$ is the unit vector indicating the direction of propagation of the wave, A_0 is its amplitude, $\mathbf{x} = (x_1, x_2, x_3)$ is the radius vector of an arbitrary point of space drawn from the point O, (,) is scalar multiplication, and k is the wave number, which is assumed to real and positive. The scattered field $u_p(\mathbf{x})$ satisfies the Helmholtz equation in the exterior D of the body D_*, and the boundary condition

$$(\Delta + k^2)\, u_p(\mathbf{x}) = 0;\ \ du_p/dn = -du_n/dn,\ \ \mathbf{x} \in S \tag{2}$$

and also the Sommerfeld radiation condition, which can be written in the form

$$u_p(\mathbf{x}) = -4\pi\,|\mathbf{x}|^{-1} \exp(ik\,|\mathbf{x}|)\, f(k; \mathbf{l}, \mathbf{v}) + o(|\mathbf{x}|^{-1})$$
$$(|\mathbf{x}| \to \infty)$$

Here Δ is the Laplace operator, d/dn is the derivative with respect to the direction of the external normal to S, $|\mathbf{x}| = (\mathbf{x}, \mathbf{x})^{1/2}$ is the length of the vector $\mathbf{x}$, $\mathbf{v} = \mathbf{x}\,|\mathbf{x}|^{-1}$ is the unit vector in the direction x, and $f(k; \mathbf{l}, \mathbf{v})$ is the scattering amplitude.

Consider the problem of determining the function $\varepsilon F(t, \varphi)$ from the scattering amplitude known in one of the directions in space, specified by a discrete set of wave numbers.

The solution of the direct problem (1), (2) is unique and can be represented for $x \in S$ as the solution of the integral equation /2/

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